

Translating Equality Downwards

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Abstract

Downward translation of equality refers to cases where a collapse of some pair of complexity classes would induce a collapse of some other pair of complexity classes that (a priori) one expects are smaller. Recently, the first downward translation of equality was obtained that applied to the polynomial hierarchy—in particular, to bounded access to its levels [HHH97]. In this paper, we provide a much broader downward translation that extends not only that downward translation but also that translation’s elegant enhancement by Buhrman and Fortnow [BF96]. Our work also sheds light on previous research on the structure of refined polynomial hierarchies [Sel95, Sel94], and strengthens the connection between the collapse of bounded query hierarchies and the collapse of the polynomial hierarchy.

1 Introduction

Does the collapse of low-complexity classes imply the collapse of higher-complexity classes? Does the collapse of high-complexity classes imply the collapse of lower-complexity classes? These questions—known respectively as downward and upward translation of equality—have long been central topics in computational complexity theory. For example, in the seminal paper on the polynomial hierarchy, Meyer and Stockmeyer [MS72] proved that the polynomial hierarchy displays upward translation of equality (e.g., $P = NP \Rightarrow P = PH$).

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The issue of whether the polynomial hierarchy—its levels and/or bounded access to its levels—ever displays *downward* translation of equality has proved more difficult. The first such result was recently obtained by Hemaspaandra, Hemaspaandra, and Hempel [HHH97], who proved that if for some high level of the polynomial hierarchy one query equals two queries, then the hierarchy collapses down not just to one query to that level, but rather to that level itself. That is, they proved the following result (note: the levels of the polynomial hierarchy [MS72, Sto77] are denoted in the standard way, namely, $\Sigma_0^p = P$, $\Sigma_1^p = NP$, $\Sigma_k^p = NP^{\Sigma_{k-1}^p}$ for each $k > 1$, and $\Pi_k^p = \{L \mid \overline{L} \in \Sigma_k^p\}$ for each $k \geq 0$).

Theorem 1.1 ([HHH97]) *For each $k > 2$: If $P^{\Sigma_k^p[1]} = P^{\Sigma_k^p[2]}$, then $\Sigma_k^p = \Pi_k^p = PH$.*

This theorem has two clear directions in which one might hope to strengthen it. First, one might ask not just about one-versus-two queries but rather about j -versus- $j + 1$ queries. Second, one might ask if the $k > 2$ can be improved to $k > 1$. Both of these have been achieved. The first strengthening was achieved in a more technical section of the same paper by Hemaspaandra, Hemaspaandra, and Hempel [HHH97]. They showed that Theorem 1.1 was just the $j = 1$ special case of a more general downward translation result they established, for $k > 2$, between bounded access to Σ_k^p and the boolean hierarchy over Σ_k^p . The second type of strengthening was achieved by Buhrman and Fortnow [BF96], who in a very elegant paper showed that Theorem 1.1 holds even for $k = 2$, but who also showed that no relativizable technique can establish Theorem 1.1 for $k = 1$.

Neither of the results or proofs just mentioned is broad enough to achieve both strengthenings simultaneously. In this paper we present new results strong enough to achieve this—and more. In particular, we unify and extend all the above results, and also unify with these results and extend the most computer-science-relevant portions of the work of Selivanov ([Sel95, Section 8], [Sel94]) on whether refined polynomial hierarchy classes are closed under complementation.

To explain exactly what we do and how it extends previous results, we now state the above-mentioned results in the more general forms in which they were actually established, though in some cases with different notations or statements (see, e.g., the interesting recent paper of Wagner [Wag97] regarding the relationship between “delta notation” and truth-table classes). Before stating the results, we must very briefly remind the reader of three definitions/notations, namely of the Δ levels of the polynomial hierarchy, of symmetric difference, and of boolean hierarchies.

Definition 1.2 1. (see [MS72]) *As is standard, for each $k \geq 1$, Δ_k^p denotes $P^{\Sigma_{k-1}^p}$.*

2. *For any classes \mathcal{C} and \mathcal{D} ,*

$$\mathcal{C}\Delta\mathcal{D} = \{L \mid (\exists C \in \mathcal{C})(\exists D \in \mathcal{D})[L = C\Delta D]\},$$

where $C\Delta D = (C - D) \cup (D - C)$.

3. ([CGH⁺88, CGH⁺89], see also [Hau14, KSW87]) *Let \mathcal{C} be any complexity class. We now define the levels of the boolean hierarchy.*

(a) $\text{DIFF}_1(\mathcal{C}) = \mathcal{C}$.

(b) *For any $k \geq 1$, $\text{DIFF}_{k+1}(\mathcal{C}) = \{L \mid (\exists L_1 \in \mathcal{C})(\exists L_2 \in \text{DIFF}_k(\mathcal{C}))[L = L_1 - L_2]\}$.*

- (c) For any $k \geq 1$, $\text{coDIFF}_k(\mathcal{C}) = \{L \mid \bar{L} \in \text{DIFF}_k(\mathcal{C})\}$.
- (d) $\text{BH}(\mathcal{C})$, the boolean hierarchy over \mathcal{C} , is $\bigcup_{k \geq 1} \text{DIFF}_k$.

The relationship between the levels of the boolean hierarchy over Σ_k^p and bounded access to Σ_k^p is as follows. For each $k \geq 0$ and each $m \geq 0$, $\text{P}^{\Sigma_k^p[m]} \subseteq \text{DIFF}_{m+1}(\Sigma_k^p) \subseteq \text{P}^{\Sigma_k^p[m+1]}$.
 $\subseteq \text{coDIFF}_{m+1}(\Sigma_k^p) \subseteq \text{P}^{\Sigma_k^p[m+1]}$.

Now we can state what the earlier papers achieved (and, in doing so, those papers obtained as corollaries the results mentioned above).

Theorem 1.3 1. ([HHH97]) Let $m > 0$, $0 \leq i < j < k$, and $i < k - 2$. If $\text{P}^{\Sigma_i^p[1]} \Delta \text{DIFF}_m(\Sigma_k^p) = \text{P}^{\Sigma_j^p[1]} \Delta \text{DIFF}_m(\Sigma_k^p)$, then $\text{DIFF}_m(\Sigma_k^p) = \text{coDIFF}_m(\Sigma_k^p)$.

2. ([BF96]) If $\text{P} \Delta \Sigma_2^p = \text{NP} \Delta \Sigma_2^p$, then $\Sigma_2^p = \Pi_2^p = \text{PH}$.

3. ([Sel95, Sel94]) If $\Sigma_i^p \Delta \Sigma_k^p$ is closed under complementation, then the polynomial hierarchy collapses.¹

In this paper, we unify all three of the above results—and achieve the strengthened corollary alluded to above (and stated later as Corollary 4.1) regarding the relative power of j and $j + 1$ queries to Σ_k^p —by proving the following two results, each of which is a downward translation of equality.

1. Let $m > 0$ and $0 < i < k$. If $\Sigma_i^p \Delta \text{DIFF}_m(\Sigma_k^p) = \Sigma_i^p \Delta \text{DIFF}_m(\Sigma_k^p)$, then $\text{DIFF}_m(\Sigma_k^p) = \text{coDIFF}_m(\Sigma_k^p)$.
2. Let $m > 0$ and $0 < i < k - 1$. If $\Sigma_i^p \Delta \text{DIFF}_m(\Sigma_k^p)$ is closed under complementation, then $\text{DIFF}_m(\Sigma_k^p) = \text{coDIFF}_m(\Sigma_k^p)$.

Informally put, the technical innovation of our proof is as follows. In the previous work extending Theorem 1.1 to the boolean hierarchy (part 1 of Theorem 1.3), the “coordination” difficulties presented by the fact that boolean hierarchy sets are in effect handled via collections of machines were resolved via using certain lexicographically extreme objects as clear signposts to signal machines with. In the current stronger context that approach fails. Instead, we integrate into the structure of easy-hard-technique proofs (especially those of [HHH97, BF96]) the so-called “telescoping” normal form possessed by the boolean hierarchy over Σ_k^p (for each k) [CGH⁺88, Hau14, Wec85], which in concept dates back to Hausdorff’s work on algebras of sets. This normal form guarantees that if $L \in \text{DIFF}_m(\Sigma_k^p)$, then there are sets $L_1, L_2, \dots, L_m \in \Sigma_k^p$ such that $L_{\text{DIFF}_m(\Sigma_k^p)} = L_1 - (L_2 - (L_3 - \dots (L_{m-1} - L_m) \dots))$ and $L_1 \supseteq L_2 \supseteq \dots \supseteq L_{m-1} \supseteq L_m$. (Picture, if you will, an archery target with concentric rings of membership and nonmembership. That is exactly the effect created by this normal form.)

As noted at the end of Section 4, the stronger downward translations we obtain yield a strengthened collapse of the polynomial hierarchy under the assumption of a collapse in the bounded query hierarchy over NP^{NP} .

¹Selivanov [Sel95, Sel94] establishes only that the hierarchy collapses to a higher level, namely a level that contains Σ_{k+1}^p ; thus this result is an upward translation of equality rather than a downward translation of equality.

We conclude this section with some additional literature pointers. We mention that the proofs of Theorem 1.1 and all that grew out of it—including this paper—are indebted to, and use extensions of, the “easy-hard” technique that was invented by Kadin ([Kad88], as further developed in [Wag87, Wag89, BCO93, CK96]) to study *upward* translations of equality resulting from the collapse of the boolean hierarchy. We also mention that there is a body of literature showing that equality of exponential-time classes translates downwards in a limited sense: Relationships are obtained with whether *sparse* sets collapse within lower time classes (the classic paper in this area is that of Hartmanis, Immerman, and Sewelson [HIS85], see also [RRW94]; limitations of such results are presented in [All91, AW90, HJ95]). Other than being a restricted type of downward translation of equality, that body of work has no close connection with the present paper due to that body of work’s applicability only to sparse sets.

2 Main Result: A New Downward Translation of Equality

We first need a definition and a useful lemma.

Definition 2.1 *For any sets C and D :*

$$C\tilde{\Delta}D = \{\langle x, y \rangle \mid x \in C \Leftrightarrow y \notin D\}.$$

Lemma 2.2 *C is \leq_m^p -complete for \mathcal{C} and D is \leq_m^p -complete for \mathcal{D} , then $C\tilde{\Delta}D$ is \leq_m^p -hard for $\mathcal{C}\Delta\mathcal{D}$.*

Proof: Let $L \in \mathcal{C}\Delta\mathcal{D}$. We need to show that $L \leq_m^p C\tilde{\Delta}D$. Let $\hat{C} \in \mathcal{C}$ and $\hat{D} \in \mathcal{D}$ be such that $L = \hat{C}\Delta\hat{D}$. Let $\hat{C} \leq_m^p C$ by f_C , and $\hat{D} \leq_m^p D$ by f_D . Then $x \in L$ iff $x \in \hat{C}\Delta\hat{D}$, $x \in \hat{C}\Delta\hat{D}$ iff $(x \in \hat{C} \Leftrightarrow x \notin \hat{D})$, $(x \in \hat{C} \Leftrightarrow x \notin \hat{D})$ iff $(f_C(x) \in C \Leftrightarrow f_D(x) \notin D)$, and $(f_C(x) \in C \Leftrightarrow f_D(x) \notin D)$ iff $\langle f_C(x), f_D(x) \rangle \in C\tilde{\Delta}D$. ■

We now state our main result.

Theorem 2.3 *Let $m > 0$ and $0 < i < k$. If $\Delta_i^p \Delta \text{DIFF}_m(\Sigma_k^p) = \Sigma_i^p \Delta \text{DIFF}_m(\Sigma_k^p)$, then $\text{DIFF}_m(\Sigma_k^p) = \text{coDIFF}_m(\Sigma_k^p)$.*

This result almost follows from the forthcoming Theorem 3.1—or, to be more accurate, almost all of its cases are easy corollaries of Theorem 3.1. However, the remaining cases—which are the most challenging ones—also need to be established, and Theorem 2.4 does exactly that.

Theorem 2.4 *Let $m > 0$ and $k > 1$. If $\Delta_{k-1}^p \Delta \text{DIFF}_m(\Sigma_k^p) = \Sigma_{k-1}^p \Delta \text{DIFF}_m(\Sigma_k^p)$, then $\text{DIFF}_m(\Sigma_k^p) = \text{coDIFF}_m(\Sigma_k^p)$.*

Definition 2.5 *For each $k > 1$, choose any fixed problem that is \leq_m^p -complete for Σ_k^p and call it $L'_{\Sigma_k^p}$. Now, having fixed such sets, for each $k > 1$ choose one fixed set $L''_{\Sigma_{k-2}^p}$ that is in Σ_{k-2}^p and*

one fixed set $L_{\Sigma_{k-1}^p}$ that is \leq_m^p -complete for Σ_{k-1}^p and that satisfy²

$$L'_{\Sigma_k^p} = \{x \mid (\exists y \in \Sigma^{|x|})(\forall z \in \Sigma^{|x|})[\langle x, y, z \rangle \in L''_{\Sigma_{k-2}^p}]\}$$

and

$$L_{\Sigma_{k-1}^p} = \{\langle x, y, z \rangle \mid |x| = |y| \wedge (\exists z')[(|x| = |y| = |z'|) \wedge \langle x, y, z z' \rangle \notin L''_{\Sigma_{k-2}^p}]\}.$$

Proof of Theorem 2.4 Let $L_{\Sigma_{k-1}^p} \in \Sigma_{k-1}^p$ be as defined in Definition 2.5, and let $L_{\Delta_{k-1}^p}$ and $L_{\text{DIFF}_m(\Sigma_k^p)}$ be any fixed \leq_m^p -complete sets for Δ_{k-1}^p and $\text{DIFF}_m(\Sigma_k^p)$, respectively; such languages exist, e.g., via the standard canonical complete set constructions using enumerations of clocked machines. From Lemma 2.2 it follows that $L_{\Delta_{k-1}^p} \tilde{\Delta} L_{\text{DIFF}_m(\Sigma_k^p)}$ is \leq_m^p -hard for $\Delta_{k-1}^p \Delta \text{DIFF}_m(\Sigma_k^p)$. (Though this is not needed for this proof, we note in passing that it also can be easily seen to be in $\Delta_{k-1}^p \Delta \text{DIFF}_m(\Sigma_k^p)$, and so it is in fact \leq_m^p -complete for $\Delta_{k-1}^p \Delta \text{DIFF}_m(\Sigma_k^p)$.) Since $L_{\Sigma_{k-1}^p} \tilde{\Delta} L_{\text{DIFF}_m(\Sigma_k^p)} \in \Sigma_{k-1}^p \Delta \text{DIFF}_m(\Sigma_k^p)$ and by assumption $\Delta_{k-1}^p \Delta \text{DIFF}_m(\Sigma_k^p) = \Sigma_{k-1}^p \Delta \text{DIFF}_m(\Sigma_k^p)$, there exists a polynomial-time many-one reduction h from $L_{\Sigma_{k-1}^p} \tilde{\Delta} L_{\text{DIFF}_m(\Sigma_k^p)}$ to $L_{\Delta_{k-1}^p} \tilde{\Delta} L_{\text{DIFF}_m(\Sigma_k^p)}$ (in light of the latter's \leq_m^p -hardness). So, for all $x_1, x_2 \in \Sigma^*$: if $h(\langle x_1, x_2 \rangle) = \langle y_1, y_2 \rangle$, then $(x_1 \in L_{\Sigma_{k-1}^p} \Leftrightarrow x_2 \notin L_{\text{DIFF}_m(\Sigma_k^p)})$ if and only if $(y_1 \in L_{\Delta_{k-1}^p} \Leftrightarrow y_2 \notin L_{\text{DIFF}_m(\Sigma_k^p)})$. Equivalently, for all $x_1, x_2 \in \Sigma^*$:

if $h(\langle x_1, x_2 \rangle) = \langle y_1, y_2 \rangle$,
then

$$(x_1 \in L_{\Sigma_{k-1}^p} \Leftrightarrow x_2 \in L_{\text{DIFF}_m(\Sigma_k^p)}) \text{ if and only if } (y_1 \in L_{\Delta_{k-1}^p} \Leftrightarrow y_2 \in L_{\text{DIFF}_m(\Sigma_k^p)}). \quad (**)$$

We can use h to recognize some of $\overline{L_{\text{DIFF}_m(\Sigma_k^p)}}$ by a $\text{DIFF}_m(\Sigma_k^p)$ algorithm. In particular, we say that a string x is *easy for length n* if there exists a string x_1 such that $|x_1| \leq n$ and $(x_1 \in L_{\Sigma_{k-1}^p} \Leftrightarrow y_1 \notin L_{\Delta_{k-1}^p})$ where $h(\langle x_1, x \rangle) = \langle y_1, y_2 \rangle$.

Let p be a fixed polynomial, which will be exactly specified later in the proof. We have the following algorithm to test whether $x \in \overline{L_{\text{DIFF}_m(\Sigma_k^p)}}$ in the case that (our input) x is an easy string for $p(|x|)$. Guess x_1 with $|x_1| \leq p(|x|)$, let $h(\langle x_1, x \rangle) = \langle y_1, y_2 \rangle$, and accept if and only if $(x_1 \in L_{\Sigma_{k-1}^p} \Leftrightarrow y_1 \notin L_{\Delta_{k-1}^p})$ and $y_2 \in L_{\text{DIFF}_m(\Sigma_k^p)}$.³ This algorithm is not necessarily a $\text{DIFF}_m(\Sigma_k^p)$ algorithm, but it does inspire the following $\text{DIFF}_m(\Sigma_k^p)$ algorithm to test whether $x \in \overline{L_{\text{DIFF}_m(\Sigma_k^p)}}$ in the case that x is an easy string for $p(|x|)$.

Let L_1, L_2, \dots, L_m be languages in Σ_k^p such that $L_{\text{DIFF}_m(\Sigma_k^p)} = L_1 - (L_2 - (L_3 - \dots (L_{m-1} - L_m) \dots))$ and $L_1 \supseteq L_2 \supseteq \dots \supseteq L_{m-1} \supseteq L_m$ (this can be done, as it is simply the “telescoping” normal form of the levels of the boolean hierarchy over Σ_k^p , see [CGH⁺88, Hau14, Wec85]). For

²By Stockmeyer’s [Sto77] standard quantifier characterization of the polynomial hierarchy’s levels, there do exist sets satisfying this definition.

³To understand what is going on here, simply note that if $(x_1 \in L_{\Sigma_{k-1}^p} \Leftrightarrow y_1 \notin L_{\Delta_{k-1}^p})$ holds then by equation (**) we have $x \in \overline{L_{\text{DIFF}_m(\Sigma_k^p)}} \Leftrightarrow y_2 \in L_{\text{DIFF}_m(\Sigma_k^p)}$. Note also that both of $x_1 \in L_{\Sigma_{k-1}^p}$ and $y_1 \notin L_{\Delta_{k-1}^p}$ can be very easily tested by a machine that has a Σ_{k-1}^p oracle.

$1 \leq r \leq m$, define L'_r as the language accepted by the following Σ_k^p machine: On input x , guess x_1 with $|x_1| \leq p(|x|)$, let $h(\langle x_1, x \rangle) = \langle y_1, y_2 \rangle$, and accept if and only if $(x_1 \in L_{\Sigma_{k-1}^p} \Leftrightarrow y_1 \notin L_{\Delta_{k-1}^p})$ and $y_2 \in L_r$.

Note that $L'_r \in \Sigma_k^p$ for each r , and that $L'_1 \supseteq L'_2 \supseteq \dots \supseteq L'_{m-1} \supseteq L'_m$. We will show that if x is an easy string for length $p(|x|)$, then $x \in \overline{L_{\text{DIFF}_m(\Sigma_k^p)}}$ if and only if $x \in L'_1 - (L'_2 - (L'_3 - \dots (L'_{m-1} - L'_m) \dots))$.

So suppose that x is an easy string for $p(|x|)$. Define r' to be the unique integer such that (a) $0 \leq r' \leq m$, (b) $x \in L'_s$ for $1 \leq s \leq r'$, and (c) $x \notin L'_s$ for $s > r'$. It is immediate that $x \in L'_1 - (L'_2 - (L'_3 - \dots (L'_{m-1} - L'_m) \dots))$ if and only if r' is odd.

Let w be some string such that:

- $(\exists x_1 \in (\Sigma^*)^{\leq p(|x|)})(\exists y_1)[h(\langle x_1, x \rangle) = \langle y_1, w \rangle \wedge (x_1 \in L_{\Sigma_{k-1}^p} \Leftrightarrow y_1 \notin L_{\Delta_{k-1}^p})]$, and
- $w \in L_{r'}$ if $r' > 0$.

Note that such a w exists, since x is easy for $p(|x|)$. By the definition of r' (namely, since $x \notin L'_s$ for $s > r'$), $w \notin L_s$ for all $s > r'$. It follows that $w \in L_{\text{DIFF}_m(\Sigma_k^p)}$ if and only if r' is odd.

It is clear, keeping in mind the definition of h , that $x \in \overline{L_{\text{DIFF}_m(\Sigma_k^p)}}$ iff $w \in L_{\text{DIFF}_m(\Sigma_k^p)}$, $w \in L_{\text{DIFF}_m(\Sigma_k^p)}$ iff r' is odd, and r' is odd iff $x \in L'_1 - (L'_2 - (L'_3 - \dots (L'_{m-1} - L'_m) \dots))$. This completes the case where x is easy, as $L'_1 - (L'_2 - (L'_3 - \dots (L'_{m-1} - L'_m) \dots))$ in effect specifies a $\text{DIFF}_m(\Sigma_k^p)$ algorithm.

We say that x is *hard for length n* if $|x| \leq n$ and x is not easy for length n , i.e., if $|x| \leq n$ and for all x_1 with $|x_1| \leq n$, $(x_1 \in L_{\Sigma_{k-1}^p} \Leftrightarrow y_1 \in L_{\Delta_{k-1}^p})$, where $h(\langle x_1, x \rangle) = \langle y_1, y_2 \rangle$. Note that if x is hard for $p(|x|)$, then $x \notin L'_1$.

If x is a hard string for length $p(|x|)$, then x induces a many-one reduction from $(L_{\Sigma_{k-1}^p})^{\leq p(|x|)}$ to $L_{\Delta_{k-1}^p}$, namely, $f(x_1) = y_1$, where $h(\langle x_1, x \rangle) = \langle y_1, y_2 \rangle$. (Note that f is computable in time polynomial in $\max(|x|, |x_1|)$.) So it is not hard to see that if we choose p appropriately large, then a hard string x for $p(|x|)$ induces Σ_{k-1}^p algorithms for $(L_1)^{=|x|}, (L_2)^{=|x|}, \dots, (L_m)^{=|x|}$ (essentially since each is in $\Sigma_k^p = \text{NP}^{\Sigma_{k-1}^p}$, $L_{\Sigma_{k-1}^p}$ is \leq_m^p -complete for Σ_{k-1}^p , and $\text{NP}^{\Delta_{k-1}^p} = \Sigma_{k-1}^p$), which we can use to obtain a $\text{DIFF}_m(\Sigma_{k-1}^p)$ algorithm for $L_{\text{DIFF}_m(\Sigma_k^p)}$, and thus certainly a $\text{DIFF}_m(\Sigma_k^p)$ algorithm for $(\overline{L_{\text{DIFF}_m(\Sigma_k^p)}})^{=|x|}$.

However, there is a problem. The problem is that we cannot combine the $\text{DIFF}_m(\Sigma_k^p)$ algorithms for easy and hard strings into one $\text{DIFF}_m(\Sigma_k^p)$ algorithm for $\overline{L_{\text{DIFF}_m(\Sigma_k^p)}}$ that works all strings. Why? It is too difficult to decide whether a string is easy or hard; to decide this deterministically takes one query to Σ_k^p , and we cannot do that in a $\text{DIFF}_m(\Sigma_k^p)$ algorithm. This is also the reason why the methods from [HHH97] failed to prove that if $\text{P}\Delta\Sigma_2^p = \text{NP}\Delta\Sigma_2^p$, then $\Sigma_2^p = \Pi_2^p$. Recall from the introduction that the latter theorem was proven by Buhrman and Fortnow [BF96]. We will use their technique at this point. The following lemma, which we will prove after we have finished the proof of this theorem, states a generalized version of the technique from [BF96]. It has been generalized to deal with arbitrary levels of the polynomial hierarchy and to be useful in settings involving boolean hierarchies.

Lemma 2.6 *Let $k > 1$. For all $L \in \Sigma_k^p$, there exist a polynomial q and a set $\widehat{L} \in \Pi_{k-1}^p$ such that*

1. *for each natural number n' , $q(n') \geq n'$,*
2. *$\widehat{L} \subseteq \overline{L}$, and*
3. *if x is hard for $q(|x|)$, then $x \in \overline{L}$ iff $x \in \widehat{L}$.*

We defer the proof of Lemma 2.6 until later in the paper, and we now continue with the proof of the current theorem. From Lemma 2.6, it follows that there exist sets $\widehat{L}_1, \widehat{L}_2, \dots, \widehat{L}_m \in \Pi_{k-1}^p$ and polynomials q_1, q_2, \dots, q_m with the following properties for all $1 \leq r \leq m$:

1. $\widehat{L}_r \subseteq \overline{L_r}$, and
2. if x is hard for $q_r(|x|)$, then $x \in \overline{L_r}$ iff $x \in \widehat{L}_r$.

Take p to be an (easy-to-compute—we may without loss of generality require that there is an ℓ such that it is of the form $n^\ell + \ell$) polynomial such that p is at least as large as all the q_r s, i.e., such that, for each natural number n' , we have $p(n') \geq \max\{q_1(n'), \dots, q_m(n')\}$. By the definition of hardness and condition 1 of Lemma 2.6, if x is hard for $p(|x|)$ then x is hard for $q_r(|x|)$ for all $1 \leq r \leq m$. As promised earlier, we have now specified p . Define $\widehat{L}_{\text{DIFF}_m(\Sigma_k^p)}$ as follows: On input x , guess r , r even, $0 \leq r \leq m$, and accept if and only if

- $x \in L_r$ or $r = 0$, and
- if $r < m$, then $x \in \widehat{L}_{r+1}$.

Clearly, $\widehat{L}_{\text{DIFF}_m(\Sigma_k^p)} \in \Pi_k^p$. In addition, this set inherits certain properties from the \widehat{L}_r s. In particular, in light of the definition of $\widehat{L}_{\text{DIFF}_m(\Sigma_k^p)}$, the definitions of the \widehat{L}_r s, and the fact that

$$x \in \overline{\widehat{L}_{\text{DIFF}_m(\Sigma_k^p)}} \text{ iff for some even } r, 0 \leq r \leq m, \text{ we have: } (x \in L_r \text{ or } r = 0) \text{ and } (x \in \overline{L_{r+1}} \text{ or } r = m),$$

we have that the following properties hold:

1. $\widehat{L}_{\text{DIFF}_m(\Sigma_k^p)} \subseteq \overline{\widehat{L}_{\text{DIFF}_m(\Sigma_k^p)}}$, and
2. if x is hard for $p(|x|)$, then $x \in \overline{\widehat{L}_{\text{DIFF}_m(\Sigma_k^p)}}$ iff $x \in \widehat{L}_{\text{DIFF}_m(\Sigma_k^p)}$.

Finally, we are ready to give the algorithm. Recall that L'_1, L'_2, \dots, L'_m are sets in Σ_k^p such that:

1. $L'_1 \supseteq L'_2 \supseteq \dots \supseteq L'_{m-1} \supseteq L'_m$, and
2. if x is easy for $p(|x|)$, then $x \in \overline{\widehat{L}_{\text{DIFF}_m(\Sigma_k^p)}}$ if and only if $x \in L'_1 - (L'_2 - (L'_3 - \dots (L'_{m-1} - L'_m) \dots))$, and
3. if x is hard for $p(|x|)$, then $x \notin L'_1$.

We claim that for all x , $x \in \overline{\widehat{L}_{\text{DIFF}_m(\Sigma_k^p)}}$ iff $x \in (L'_1 \cup \widehat{L}_{\text{DIFF}_m(\Sigma_k^p)}) - (L'_2 - (L'_3 - \dots (L'_{m-1} - L'_m) \dots))$, which completes the proof of Theorem 2.4, as Σ_k^p is closed under union.

(\Rightarrow) If x is easy for $p(|x|)$, then $x \in L'_1 - (L'_2 - (L'_3 - \dots (L'_{m-1} - L'_m) \dots))$, and so certainly $x \in (L'_1 \cup \widehat{L}_{\text{DIFF}_m(\Sigma_k^p)}) - (L'_2 - (L'_3 - \dots (L'_{m-1} - L'_m) \dots))$. If x is hard for $p(|x|)$, then

$x \in \widehat{L}_{\text{DIFF}_m(\Sigma_k^p)}$ and $x \notin L'_r$ for all r (since $x \notin L'_1$ and $L'_1 \supseteq L'_2 \supseteq \dots$). Thus, $x \in (L'_1 \cup \widehat{L}_{\text{DIFF}_m(\Sigma_k^p)}) - (L'_2 - (L'_3 - \dots (L'_{m-1} - L'_m) \dots))$.

(\Leftarrow) Suppose $x \in (L'_1 \cup \widehat{L}_{\text{DIFF}_m(\Sigma_k^p)}) - (L'_2 - (L'_3 - \dots (L'_{m-1} - L'_m) \dots))$. If $x \in \widehat{L}_{\text{DIFF}_m(\Sigma_k^p)}$, then $x \in \overline{L_{\text{DIFF}_m(\Sigma_k^p)}}$. If $x \notin \widehat{L}_{\text{DIFF}_m(\Sigma_k^p)}$, then $x \in L'_1 - (L'_2 - (L'_3 - \dots (L'_{m-1} - L'_m) \dots))$ and so x must be easy for $p(|x|)$ (as $x \in L'_1$, and this is possible only if x is easy for $p(|x|)$). However, this says that $x \in \overline{L_{\text{DIFF}_m(\Sigma_k^p)}}$. \blacksquare

Having completed the proof of the theorem, we now return to the deferred proof of the lemma used within the theorem.

Proof of Lemma 2.6. Let $L \in \Sigma_k^p$. We need to show that there exist a polynomial q and a set $\widehat{L} \in \Pi_{k-1}^p$ such that

1. $\widehat{L} \subseteq \overline{L}$, and
2. if x is hard for $q(|x|)$, then $x \in \overline{L}$ iff $x \in \widehat{L}$.

From Definition 2.5, we know that: $L'_{\Sigma_k^p}$ is \leq_m^p -complete for Σ_k^p , $L_{\Sigma_{k-1}^p} \in \Sigma_{k-1}^p$, $L''_{\Sigma_{k-2}^p} \in \Sigma_{k-2}^p$, and

1. $L_{\Sigma_{k-1}^p} = \{\langle x, y, z \rangle \mid |x| = |y| \wedge (\exists z')[(|x| = |y| = |zz'|) \wedge \langle x, y, zz' \rangle \notin L''_{\Sigma_{k-2}^p}]\}$, and
2. $L'_{\Sigma_k^p} = \{x \mid (\exists y \in \Sigma^{|x|})(\forall z \in \Sigma^{|x|})[\langle x, y, z \rangle \in L''_{\Sigma_{k-2}^p}]\}$.

Note that $\overline{L'_{\Sigma_k^p}} = \{x \mid (\forall y \in \Sigma^{|x|})(\exists z \in \Sigma^{|x|})[\langle x, y, z \rangle \notin L''_{\Sigma_{k-2}^p}]\}$.

Since $L \in \Sigma_k^p$, and $L'_{\Sigma_k^p}$ is \leq_m^p -complete for Σ_k^p , there exists a polynomial-time computable function g such that, for all x , $x \in L$ iff $g(x) \in L'_{\Sigma_k^p}$.

Let q be such that (a) $(\forall x \in \Sigma^n)(\forall y \in \Sigma^{|g(x)|})(\forall z \in (\Sigma^*)^{\leq |g(x)|})[q(n) \geq |\langle g(x), y, z \rangle|]$ and (b) $(\forall \widehat{m} \geq 0)[q(\widehat{m} + 1) > q(\widehat{m}) > 0]$. Note that we have ensured that for each natural number n' , $q(n') \geq n'$.

If x is a hard string for length $q(|x|)$, then x induces a many-one reduction from $(L_{\Sigma_{k-1}^p})^{\leq q(|x|)}$ to $L_{\Delta_{k-1}^p}$, namely, $f_x(x_1) = y_1$, where $h(\langle x_1, x \rangle) = \langle y_1, y_2 \rangle$. (This is the h from the proof of Theorem 2.4. One should treat the current proof as if it occurs immediately after the statement of Lemma 2.6.) Note that f_x is computable in time polynomial in $\max(|x|, |x_1|)$.

Let \widehat{L} be the language accepted by the following Π_{k-1}^p machine:⁴

On input x :

Compute $g(x)$

Guess y such that $|y| = |g(x)|$

⁴For $k > 1$, $\Pi_{k-1}^p = \text{coNP}^{\Sigma_{k-2}^p}$, and by a Π_{k-1}^p machine we mean a co-nondeterministic machine with a Σ_{k-2}^p oracle. A co-nondeterministic machine by definition accepts iff *all* of its computation paths are accepting paths. (Some authors prefer requiring that all paths be rejecting paths; the definitions are equivalent as long as one is consistent throughout regarding which model one is using.)

Set $w = \epsilon$ (i.e., the empty string)

While $|w| < |g(x)|$

if the Δ_{k-1}^p algorithm induced by x for $L_{\Sigma_{k-1}^p}$ accepts $\langle g(x), y, w0 \rangle$

(that is, if $f_x(\langle g(x), y, w0 \rangle) \in L_{\Delta_{k-1}^p}$),

then $w = w0$

else $w = w1$

Accept if and only if $\langle g(x), y, w \rangle \notin L''_{\Sigma_{k-2}^p}$.

It remains to show that \hat{L} thus defined fulfills the properties of Lemma 2.6. First note that the machine described above is clearly a Π_{k-1}^p machine. To show that $\hat{L} \subseteq \bar{L}$, suppose that $x \in \hat{L}$. Then (keeping in mind the comments of footnote 4) for every $y \in \Sigma^{|g(x)|}$, there exists a string $w \in \Sigma^{|g(x)|}$ such that $\langle g(x), y, w \rangle \notin L''_{\Sigma_{k-2}^p}$. This implies that $g(x) \in L'_{\Sigma_k^p}$, and thus that $x \in \bar{L}$.

Finally, suppose that x is hard for $q(|x|)$ and that $x \in \bar{L}$. We have to show that $x \in \hat{L}$. Since $x \in \bar{L}$, $g(x) \in L'_{\Sigma_k^p}$. So, $(\forall y \in \Sigma^{|g(x)|})(\exists z \in \Sigma^{|g(x)|})[\langle g(x), y, z \rangle \notin L''_{\Sigma_{k-2}^p}]$. Since x is hard, $(\forall y \in \Sigma^{|g(x)|})(\forall w \in (\Sigma^*)^{\leq |g(x)|})[\langle g(x), y, w \rangle \in L_{\Sigma_{k-1}^p} \Leftrightarrow f_x(\langle g(x), y, w \rangle) \in L_{\Delta_{k-1}^p}]$. It follows that the algorithm above will find, for every $y \in \Sigma^{|g(x)|}$, a witness w such that $\langle g(x), y, w \rangle \notin L''_{\Sigma_{k-2}^p}$, and thus the algorithm will accept x . \blacksquare

3 A Downward Translation of Equality for Closure under Complementation

We now state our downward translation for closure under complementation, Theorem 3.1. Theorem 3.1 in part underpins our main result, Theorem 2.3, as Theorem 3.1 is drawn on in the proof of Theorem 2.3 (see the discussion immediately after the statement of Theorem 2.3). However, Theorem 2.3 is not a corollary of Theorem 3.1; the two results are incomparable.

Theorem 3.1 *Let $m > 0$ and $0 < i < k - 1$. If $\Sigma_i^p \Delta \text{DIFF}_m(\Sigma_k^p)$ is closed under complementation, then $\text{DIFF}_m(\Sigma_k^p) = \text{coDIFF}_m(\Sigma_k^p)$.*

Proof of Theorem 3.1 Let $L_{\Sigma_i^p}$ and $L_{\text{DIFF}_m(\Sigma_k^p)}$ be \leq_m^p -complete for Σ_i^p and $\text{DIFF}_m(\Sigma_k^p)$ respectively. Since $L_{\Sigma_i^p} \tilde{\Delta} L_{\text{DIFF}_m(\Sigma_k^p)}$ is \leq_m^p -hard for $\Sigma_i^p \Delta \text{DIFF}_m(\Sigma_k^p)$ by Lemma 2.2 (in fact, it is not hard to see that it even is \leq_m^p -complete for that class) and by assumption $\Sigma_i^p \Delta \text{DIFF}_m(\Sigma_k^p)$ is closed under complementation, there exists a polynomial-time many-one reduction h from $L_{\Sigma_i^p} \tilde{\Delta} L_{\text{DIFF}_m(\Sigma_k^p)}$ to its complement. That is, for all $x_1, x_2 \in \Sigma^*$: if $h(\langle x_1, x_2 \rangle) = \langle y_1, y_2 \rangle$, then: $\langle x_1, x_2 \rangle \in L_{\Sigma_i^p} \tilde{\Delta} L_{\text{DIFF}_m(\Sigma_k^p)} \Leftrightarrow \langle y_1, y_2 \rangle \notin L_{\Sigma_i^p} \tilde{\Delta} L_{\text{DIFF}_m(\Sigma_k^p)}$. Equivalently, for all $x_1, x_2 \in \Sigma^*$:

Fact 1:

if $h(\langle x_1, x_2 \rangle) = \langle y_1, y_2 \rangle$,

then

$$(x_1 \in L_{\Sigma_i^p} \Leftrightarrow x_2 \notin L_{\text{DIFF}_m(\Sigma_k^p)}) \text{ if and only if } (y_1 \in L_{\Sigma_i^p} \Leftrightarrow y_2 \in L_{\text{DIFF}_m(\Sigma_k^p)}).$$

We can use h to recognize some of $\overline{L_{\text{DIFF}_m(\Sigma_k^p)}}$ by a $\text{DIFF}_m(\Sigma_k^p)$ algorithm. In particular, we say that a string x is *easy for length n* if there exists a string x_1 such that $|x_1| \leq n$ and $(x_1 \in L_{\Sigma_i^p} \Leftrightarrow y_1 \in L_{\Sigma_i^p})$ where $h(\langle x_1, x \rangle) = \langle y_1, y_2 \rangle$.

Let p be a fixed polynomial, which will be exactly specified later in the proof. We have the following algorithm to test whether $x \in \overline{L_{\text{DIFF}_m(\Sigma_k^p)}}$ in the case that (our input) x is an easy string for $p(|x|)$. On input x , guess x_1 with $|x_1| \leq p(|x|)$, let $h(\langle x_1, x \rangle) = \langle y_1, y_2 \rangle$, and accept if and only if $(x_1 \in L_{\Sigma_i^p} \Leftrightarrow y_1 \in L_{\Sigma_i^p})$ and $y_2 \in L_{\text{DIFF}_m(\Sigma_k^p)}$. This algorithm is not necessarily a $\text{DIFF}_m(\Sigma_k^p)$ algorithm, but in the same way as in the proof of Theorem 2.4, we can construct sets $L'_1, L'_2, \dots, L'_m \in \Sigma_k^p$ such that if x is an easy string for length $p(|x|)$, then $x \in \overline{L_{\text{DIFF}_m(\Sigma_k^p)}}$ if and only if $x \in L'_1 - (L'_2 - (L'_3 - \dots (L'_{m-1} - L'_m) \dots))$.

We say that x is *hard for length n* if $|x| \leq n$ and x is not easy for length n , i.e., if $|x| \leq n$ and, for all x_1 with $|x_1| \leq n$, $(x_1 \in L_{\Sigma_i^p} \Leftrightarrow y_1 \notin L_{\Sigma_i^p})$, where $h(\langle x_1, x \rangle) = \langle y_1, y_2 \rangle$.

If x is a hard string for length n , then x induces a many-one reduction from $(L_{\Sigma_i^p})^{\leq n}$ to $\overline{L_{\Sigma_i^p}}$, namely, $f(x_1) = y_1$, where $h(\langle x_1, x \rangle) = \langle y_1, y_2 \rangle$. Note that f is computable in time polynomial in $\max(n, |x_1|)$.

We can use hard strings to obtain a $\text{DIFF}_m(\Sigma_{k-1}^p)$ algorithm for $L_{\text{DIFF}_m(\Sigma_k^p)}$, and thus (since $\text{DIFF}_m(\Sigma_{k-1}^p) \subseteq \text{P}^{\Sigma_{k-1}^p} \subseteq \Sigma_k^p \cap \Pi_k^p$) certainly a $\text{DIFF}_m(\Sigma_k^p)$ algorithm for $\overline{L_{\text{DIFF}_m(\Sigma_k^p)}}$. Let L_1, L_2, \dots, L_m be languages in Σ_k^p such that $L_{\text{DIFF}_m(\Sigma_k^p)} = L_1 - (L_2 - (L_3 - \dots (L_{m-1} - L_m) \dots))$. For all $1 \leq r \leq m$, let M_r be a Σ_{k-i}^p machine such that M_r with oracle $L_{\Sigma_i^p}$ recognizes L_r . Let the run-time of all M_r s be bounded by polynomial p , which without loss of generality is easily computable and satisfies $(\forall \hat{m} \geq 0)[p(\hat{m} + 1) > p(\hat{m}) > 0]$ (as promised earlier, we have now specified p). Then, for all $1 \leq r \leq m$,

$$(L_r)^{=n} = \left(L \left(M_r \left(L_{\Sigma_i^p} \right)^{\leq p(n)} \right) \right)^{=n}.$$

If there exists a hard string for length $p(n)$, then that hard string induces a reduction from $(\overline{L_{\Sigma_i^p}})^{\leq p(n)}$ to $L_{\Sigma_i^p}$.

Let $L \in \Sigma_{i-1}^p$ and r be a polynomial such that r is easily computable and, for all x , $x \in L_{\Sigma_i^p}$ iff $(\exists y \in (\Sigma^*)^{\leq r(|x|)})[\langle x, y \rangle \notin L]$.

We will show that with any hard string for length $p(n)$ in hand, call it w_n , there exist Σ_{k-1}^p algorithms for $(L_1)^{=n}, (L_2)^{=n}, \dots, (L_m)^{=n}$.

Let $\widehat{M_r}$ be the following Σ_{k-i}^p machine. On input x of length n , $\widehat{M_r}(x)$ simulates the work of $M_r(x)$ until $M_r(x)$ asks a query, call it q . Then $\widehat{M_r}$ guesses whether this query will be answered “Yes” or “No.” If $\widehat{M_r}$ guesses “Yes,” then $\widehat{M_r}$ guesses a certificate $y \in (\Sigma^*)^{\leq r(|q|)}$, makes the query $\langle q, y \rangle$ to L , rejects if the answer is “Yes,” and proceeds with the simulation if the answer is “No.” If $\widehat{M_r}$ guesses that the answer to q is “No,” then $\widehat{M_r}$ guesses that $q \notin L_{\Sigma_i^p}$, or in other words that $q \in \overline{L_{\Sigma_i^p}}$. Now we will use the reduction from $\overline{L_{\Sigma_i^p}}$ to $L_{\Sigma_i^p}$, because $q \in \overline{L_{\Sigma_i^p}}$ if and only if the first component of $h(\langle q, w_n \rangle)$ is in $L_{\Sigma_i^p}$. Let q' denote the first component of $h(\langle q, w_n \rangle)$. $\widehat{M_r}$ also guesses

$y' \in (\Sigma^*)^{\leq r(|q'|)}$, makes the query $\langle q', y' \rangle$ to L , rejects if the answer is “Yes,” and proceeds with the simulation if the answer is “No.” Clearly, \widehat{M}_r is a Σ_{k-i}^p machine that recognizes $(L_r)^{=n}$ with queries to a Σ_{i-1}^p oracle, namely L .

It follows that if there exists a hard string for length $p(n)$, then this string induces a $\text{DIFF}_m(\Sigma_{k-1}^p)$ algorithm for $(L_{\text{DIFF}_m(\Sigma_k^p)})^{=n}$, and thus certainly a $\text{DIFF}_m(\Sigma_k^p)$ algorithm for $(\overline{L_{\text{DIFF}_m(\Sigma_k^p)}})^{=n}$.

It follows that there exist m Σ_k^p sets, say, \widehat{L}_r for $1 \leq r \leq m$, such that the following holds: For all x , if x (functioning as $w_{|x|}$ above) is a hard string for length $p(|x|)$, then $x \in \overline{L_{\text{DIFF}_m(\Sigma_k^p)}}$ if and only if $x \in \widehat{L}_1 - (\widehat{L}_2 - (\widehat{L}_3 - \cdots (L_{m-1} - \widehat{L}_m) \cdots))$.

However, now we have an outright $\text{DIFF}_m(\Sigma_k^p)$ algorithm for $\overline{L_{\text{DIFF}_m(\Sigma_k^p)}}$: For $1 \leq r \leq m$ define a $\text{NP}^{\Sigma_{k-1}^p}$ machine N_r as follows: On input x , the NP base machine of N_r executes the following algorithm:

1. Using its Σ_{k-1}^p oracle, it deterministically determines whether the input x is an easy string for length $p(|x|)$. This can be done, as checking whether the input is an easy string for length $p(|x|)$ can be done by two queries to Σ_{i+1}^p , and $i+1 \leq k-1$ by our $i < k-1$ hypothesis.
2. If the previous step determined that the input is not an easy string, then the input must be a hard string for length $p(|x|)$. So simulate the Σ_k^p algorithm for \widehat{L}_r induced by this hard string (i.e., the input x itself) on input x (via our NP machine itself simulating the base level of the Σ_k^p algorithm and using the NP machine's oracle to simulate the oracle queries made by the base level NP machine of the Σ_k^p algorithm being simulated).
3. If the first step determined that the input x is easy for length $p(|x|)$, then our NP machine simulates (using itself and its oracle) the Σ_k^p algorithm for L'_r on input x .

It follows that, for all x , $x \in \overline{L_{\text{DIFF}_m(\Sigma_k^p)}}$ if and only if $x \in L(N_1) - (L(N_2) - (L(N_3) - \cdots (L(N_{m-1}) - L(N_m)) \cdots))$. Since $\overline{L_{\text{DIFF}_m(\Sigma_k^p)}}$ is complete for $\text{coDIFF}_m(\Sigma_k^p)$, it follows that $\text{DIFF}_m(\Sigma_k^p) = \text{coDIFF}_m(\Sigma_k^p)$. ■

An underlying goal of this paper is to show that Theorem 1.1 holds even for $k = 2$ and the bounded query hierarchy (that is, that Corollary 4.1 holds), and Theorem 3.1 plays a central role in establishing this. We mention that—though it is in no way needed to establish Corollary 4.1, and its proof is somewhat less transparent and more technical than that of Theorem 3.1—it is possible to prove a slightly stronger version of Theorem 3.1 that removes the asymmetry in its statement: Let $s, m > 0$ and $0 < i < k-1$. If $\text{DIFF}_s(\Sigma_i^p) \Delta \text{DIFF}_m(\Sigma_k^p)$ is closed under complementation, then $\text{DIFF}_m(\Sigma_k^p) = \text{coDIFF}_m(\Sigma_k^p)$ [HHHa].

4 Conclusions

We have proven the following downward translations of equality.

1. Let $m > 0$ and $0 < i < k$. If $\Delta_i^p \Delta \text{DIFF}_m(\Sigma_k^p) = \Sigma_i^p \Delta \text{DIFF}_m(\Sigma_k^p)$, then $\text{DIFF}_m(\Sigma_k^p) = \text{coDIFF}_m(\Sigma_k^p)$.

2. Let $m > 0$ and $0 < i < k - 1$. If $\Sigma_i^p \Delta \text{DIFF}_m(\Sigma_k^p)$ is closed under complementation, then $\text{DIFF}_m(\Sigma_k^p) = \text{coDIFF}_m(\Sigma_k^p)$.

As mentioned in the introduction, these results extend the polynomial hierarchy's previously known downward translations of equality. More importantly, they show that Theorem 1.1 can be extended to all $k > 1$ cases even for each level of the bounded query hierarchy.

Corollary 4.1 *For each $m > 0$ and each $k > 1$ it holds that:*

$$\text{P}^{\Sigma_k^p[m]} = \text{P}^{\Sigma_k^p[m+1]} \Rightarrow \text{DIFF}_m(\Sigma_k^p) = \text{coDIFF}_m(\Sigma_k^p).$$

Corollary 4.1 itself has an interesting further consequence. From this corollary, it follows (for exactly the reasons discussed in [HHHb]) that for a number of previously missing cases (namely, when $m > 1$ and $k = 2$), the hypothesis $\text{P}^{\Sigma_k^p[m]} = \text{P}^{\Sigma_k^p[m+1]}$ implies that the polynomial hierarchy collapses to about one level lower in the boolean hierarchy over Σ_{k+1}^p than could be concluded from previous papers. In particular, one can now conclude that, for all cases where $m > 0$ and $k > 1$, $\text{P}^{\Sigma_k^p[m]} = \text{P}^{\Sigma_k^p[m+1]}$ implies that each set in the polynomial hierarchy can be accepted by a P machine that makes $m - 1$ truth-table queries to Σ_{k+1}^p , and that in addition makes one query to Δ_{k+1}^p (in fact, a bit more can be claimed, see [HHHc]).

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